Phys 410
Fall 2014
Lecture #16 Summary
23 October, 2014

We considered the two-body problem of two objects interacting by means of a conservative central force, with no other external forces acting. This problem eventually simplifies from that of 6 degrees of freedom (for 2 particles in three dimensions) to a single particle moving in one dimension! After peeling off the center of mass motion (at constant velocity) and jumping to the CM reference frame, the Lagrangian simplified to: $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. The total angular momentum is constant, so that the motion (namely the vectors $\vec{r}$ and $\dot{\vec{r}}$) must remain in a fixed two-dimensional plane. The Lagrangian can be written as $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$. The two Euler-Lagrange equations can be combined into a single equation for the relative coordinate: $\mu\ddot{r} = \ell^2/\mu r^3 - dU/dr$. This can be expressed in terms of the effective potential as $\mu\ddot{r} = -dU_{eff}/dr$, where $U_{eff}(r) = U(r) + \ell^2/2\mu r^2$.

Using the example of gravity for $U(r)$ we found that the effective potential (for $\ell > 0$) has a minimum at a finite value of r, diverges as r goes to zero, and approaches zero from below as r goes to infinity. We found that mechanical energy for the relative coordinate is conserved and equal to $E = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r)$. Since kinetic energy is either positive or zero, the particle must be located in a region where $E \geq U_{eff,min}$. We see that when $E > 0$ the particle has an un-bounded orbit, while when $E < 0$ it has a bounded orbit trapped between minimum and maximum values of r.

We then solved the radial equation $\mu\ddot{r} = \frac{\ell^2}{\mu r^3} + F(r)$ for inverse-square force-laws of the form $F(r) = -\gamma/r^2$, and found a solution that expressed the radial coordinate in terms of the angular polar coordinate, $r = r(\varphi)$, in which time has been eliminated. The result is $r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}$, where $c = \frac{\ell^2}{\mu\gamma}$ is a length scale and ϵ is an un-determined constant. This is the equation for the orbit of a planet around the sun, or a satellite around the earth.

Note that when the un-determined constant $\epsilon > 1$, the denominator of $r(\varphi)$ has a zero for some angle φ, and the particle is off at infinity for that angle. This is an un-bounded orbit, like those with energy $E > 0$ noted in the last lecture. When $\epsilon < 1$ the values of $r(\varphi)$ are finite for all φ, and the orbit is bounded, like those with $E < 0$ noted above. The fact that $r(\varphi + 2\pi) = r(\varphi)$ means that the orbit is closed and periodic (this is not the case for other types of force interactions such as $F(r) \sim -1/r^3$).

The orbit for $\epsilon < 1$ is an ellipse and is described by $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = \frac{c}{1-\epsilon^2}$ is the semi-major axis, $b = \frac{c}{\sqrt{1-\epsilon^2}}$ is the semi-minor axis, and $d = a\epsilon$ is the distance from the center of the ellipse to the focus (you will prove this in HW). The ratio of semi-minor to semi-major axis lengths is $b/a = \sqrt{1-\epsilon^2}$, showing that ϵ is the ellipticity of the orbit. One can also derive Kepler's third law of planetary motion relating the orbital period τ and the semi-major axis as $\tau^2 = \frac{4\pi^2}{GM_{sun}} a^3$ for the case of a planet orbiting the sun (here one assumes that the mass of the planet is much smaller than that of the sun). Finally we calculated the total mechanical energy in the center of mass frame as $E = \frac{\gamma^2 \mu}{2\ell^2}(\epsilon^2 - 1)$. This shows that orbits with $\epsilon > 1$ are un-bounded (and described by a hyperbola), and those with $\epsilon < 1$ are bounded. Orbits with $\epsilon = 1$ are parabolic.

We next started a discussion of scattering theory. In the simplest scattering experiment one has a particle or entity (the projectile) that is launched with a known energy and momentum into a target, the projectile interacts with particles in the target, and then comes out as the same particle but with a new energy and momentum. More generally, the particle could be absorbed by the target, or be transformed into one or more different particles upon exiting the target. We can measure the exiting angle of the particle using spherical coordinates, with the z-axis along the initial projectile direction and the angular coordinates θ, φ specifying the new direction. Examples of scattering experiments include Rutherford scattering and angle-resolved photoemission spectroscopy (ARPES), which is basically the photoelectric effect on steroids.

The only quantity not controlled in a typical scattering experiment is the impact parameter b of the projectile with respect to the target particle. The impact parameter is the distance of closest approach to the target particle, assuming no forces of interaction cause the projectile to change from its initial direction. Because we cannot control the impact parameter, we have to perform many experiments in which all possible values of b are employed for the incident beam of projectiles. The objective of our calculations will be to find the functional relationship between the scattering angle and the impact parameter, namely $b = b(\theta)$, or $\theta = \theta(b)$.